

# Topological equivalences for one-parameter bifurcations of maps

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## Abstract

Homeomorphisms allowing us to prove topological equivalences between one-parameter families of maps undergoing the same bifurcation are constructed in this paper. This provides a solution for a classical problem in bifurcation theory that was set out three decades ago and remained unexpectedly unpublished until now.

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## 1 Introduction

One of the fundamental problems in the study of nonlinear dynamical systems is to know if the behavior of a system changes under small perturbations. Roughly speaking, when the dynamics of a system changes, it is said that a *bifurcation* occurs. On the contrary, if no change happens, it is said that the system is *structurally stable*.

Usually, perturbations of a dynamical system are associated to variations of parameters involved in the equation(s) which define such a system (see [7] for the original definition). Since any value of the parameters defines a particular dynamical system, if we consider all together, we have a parametric family of them. When studying bifurcations of parametric families of dynamical systems, two techniques allow us to simplify this study, namely, the *center manifold theory* and the *normal forms* method. The first one provides a reduction of the dimensionality (see for example [5]), while the second one provides a simplification of the nonlinearity, providing prototypes of behavior for vast classes

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of nonlinear systems. The method of the normal forms began with Poincaré [11] and was developed by Arnold (e.g. see [1], [2] or [8]). In a few words, this method consists in constructing a simple polynomial family, named *normal form*, using the conditions that provide a bifurcation, and demonstrating that any family satisfying those conditions is topologically equivalent to this simpler polynomial family. Note that, in such a case, the dynamical behavior of any of these families can be inferred from the dynamical behavior of the normal form which is obviously simpler to be studied.

In this work, we consider local one-parameter bifurcations for maps. Hartman-Grobman's theorem establishes that to study local bifurcations in parametric families, it suffices to consider those parameter values for which the corresponding map presents a non-hyperbolic fixed point, that is, the Jacobian matrix evaluated at this fixed point has eigenvalues of modulus 1. In particular, the simplest cases appear when only an eigenvalue has modulus 1, i.e., 1 or -1. When the eigenvalue is equal to 1, three kinds of local bifurcations can appear called *fold*, *transcritical* and *pitchfork*; while when the eigenvalue is equal to -1 a bifurcation named *flip* or *period doubling* can appear. The first three ones imply the change in the number and stability of the fixed points; while the fourth one involves additionally the appearance of a 2-periodic orbit (see for instance [12]).

In [12], it is shown that under some nonzero conditions up to the third order of the derivatives, these local bifurcations of one-parameter families of maps (fold, transcritical, pitchfork, flip) appear. In [3], similar results are obtained for higher order nonzero conditions, generalizing the necessary conditions for the appearance of such bifurcations.

Although results in [3, 12] prove that the same number of fixed points (or fixed points and period-2 points in the flip case) appears with identical type of stability, the problem of proving the topological equivalence between any family verifying the same bifurcation conditions and the corresponding simplest normal form posed originally in [1] remained unpublished until now, as claimed in [9].

Actually, as done in [1] and [9] for the fold and flip bifurcations under the lowest order conditions and in [4] for the fold, transcritical, pitchfork and flip bifurcations under higher order conditions, after applying successive diffeomorphic changes of variables and re-scalings, it is possible to demonstrate the topological equivalence between a family satisfying the bifurcation conditions of a specific order and the non truncated normal form of this same order. With non truncated normal form we refer to the normal form with higher-order terms of the Taylor polynomial that are not fixed by the bifurcation conditions. Nevertheless, that the truncation of higher order terms does not affect the topological type had not been proved yet. This is not a minor question since in cases like the Neimark-Sacker bifurcation (see for instance [3]) this truncation does affect its topological type.

The present article gives a solution for this classical problem in bifurcation theory, set out more than three decades ago in [1]. As a matter of fact, we provide a complete proof of topological equivalences between any two families of maps undergoing one of the above mentioned bifurcations in a more general way than originally posed, as shown in the results and examples herein.

In this sense, our main result consists in the construction of a homeomorphism  $h$  of conjugation between any two order preserving real homeomorphisms  $f$  and  $g$  in suitable real open intervals  $J$  and  $I$  both with the same number of fixed points with the same stabilities, i.e., a homeomorphism  $h : J \rightarrow I$ , such that  $h \circ f = g \circ h$ . When the two homeomorphisms  $f$  and  $g$  are order reversing, we construct a similar conjugacy in a suitable neighborhood of the unique fixed point.

Despite the mathematical analysis focus on topological conjugations, this construction, applied to the case of one-parameter families undergoing the same kind of bifurcation, allows us to provide a solution for the mentioned classical problem in bifurcation theory.

Of course, although the results are given for bifurcations of fixed points, they also apply to periodic points when the relevant iterate of the map is considered. For simplicity, we have stated all of our results in the context of one-dimensional maps. However, they can also be used in order to draw these bifurcations in one-parameter families of  $n$ -dimensional maps or, even more generally, of Banach spaces.

The organization of this paper is as follows. In Section 2, we construct conjugacies among any two homeomorphisms having the same number and stability of fixed points. As a result, in Section 3, we show how these results in the previous section allow us to solve the problem of demonstrating the topological equivalence between any two one-parameter families undergoing the same type of bifurcation, even in a more general way than that in which it was posed originally. Finally, in Section 4 we present conclusions and future research directions.

## 2 Technique for the construction of topological conjugacies

In this section, we develop a technique to construct homeomorphisms which give us a conjugacy between two homeomorphisms  $g$  and  $f$  satisfying similar conditions concerning the number and stability of their fixed points. These homeomorphisms will allow us to demonstrate the topological equivalence between any family satisfying certain bifurcations conditions and the corresponding normal form in Section 3.

Along this paper, we adopt definitions in [9] concerning topological conjugacy and topological equivalence. When nothing else stated, the letter  $n$  is reserved for natural numbers and Greek letters for real parameters.

In this article, we consider continuous maps with at most a countable number of isolated fixed points, i.e., without accumulation points and separated by non-empty intervals. Results for non-countable fixed points or with accumulation points are not in the scope of this work. This avoids maps that are coincident with the identity map in some non-empty interval and topological neutral fixed points.

One important concept in this work is the following, that one can find in [6].

**Definition 2.1** *A fixed point  $x_F$  of a map  $g$  is said to be semi-attracting from the left (resp. from the right), if there exists  $\eta > 0$  such that for  $x \in (x_F - \eta, x_F)$  (resp.  $x \in (x_F, x_F + \eta)$ ),  $\lim_{n \rightarrow +\infty} g^n(x) = x_F$ .*

In view of this concept, the notion of a semi-repelling fixed point to the left (resp. to the right) is clear. Actually, for a fixed point  $x_F$  of an increasing homeomorphism  $g$ , it is *semi-repelling to the left* (resp. *to the right*), iff the same fixed point  $x_F$  is semi-attracting from the left (resp. right) for the inverse  $g^{-1}$ .

Notice that an attracting fixed point is semi-attracting from the left and simultaneously from the right, and the same happens for repelling fixed points that must be simultaneously semi-repelling to the left and to the right. In particular, semi-attracting fixed points of decreasing homeomorphisms are always attracting.

We will also use the term left semi-attracting (resp. left semi-repelling) and right semi-attracting (resp. right semi-repelling) to simplify the nomenclature of these concepts.

**Definition 2.2** *We shall call transverse<sup>1</sup> fixed point to any fixed one being attracting or repelling.*

For differentiable maps, every hyperbolic fixed point is transverse. But, the reciprocal is not true since, for instance, the real map  $x + x^3$  has the non-hyperbolic transverse fixed point 0.

**Definition 2.3** *We shall call mixed-stability fixed point to any fixed point being left semi-attracting and right semi-repelling or left semi-repelling and right semi-attracting.*

The above notions can be more refined, but in our work we only need the above concepts which are related to bifurcation theory in metric spaces.

## 2.1 Maps with one fixed point

Here, we study the situation of only one fixed point for each map  $g$  and  $f$ . The left and right of the fixed points are treated separately, this treatment allows us to study maps with more than one fixed point in subsection 2.3.

**Theorem 2.4** *Let  $g : I \rightarrow g(I)$  and  $f : J \rightarrow f(J)$  two increasing homeomorphisms with domains  $I = [b, x_F]$  and  $J = [a, \overline{x}_F]$  respectively. Suppose that they verify the following conditions:*

1.  $g(x) > x$ , for all  $x \in [b, x_F)$ .
2.  $f(x) > x$ , for all  $x \in [a, \overline{x}_F)$ .

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<sup>1</sup>Relative to the diagonal, i.e., the identity map.

3.  $x_F$  is the (unique) fixed point of  $g$  and  $\bar{x}_F$  is the (unique) fixed point of  $f$ .

Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : J \rightarrow I$ , not necessarily unique, such that

$$g \circ h(x) = h \circ f(x). \quad (1)$$

**Proof.** First of all, we note that both  $x_F$  and  $\bar{x}_F$  are semi-attracting from the left. Observe that, thanks to hypotheses 1 and 2,  $f(J) \subseteq J$  and  $g(I) \subseteq I$ . In fact,  $f(J) = [f(a), \bar{x}_F] \subseteq [a, \bar{x}_F] = J$  and  $g(I) = [g(b), x_F] \subseteq [b, x_F] = I$ , since  $f(a) > a$  and  $g(b) > b$  by the hypotheses. In view of this, the homeomorphism to be defined will have the domain  $J$  and the image  $I$  and in this way it can be restricted to  $f(J)$  and  $g(I)$ .

Consider  $a$ , the leftmost point of  $J$ . We will construct the homeomorphism  $h$  subject to the condition  $h(a) = b$ . In fact, this arbitrary choice allows us to deduce that there exist infinitely many different topological conjugacies for the same functions  $g$  and  $f$ . Now, we can consider (since there are infinity of them) an increasing homeomorphism  $h_0$  in the domain  $D_0 = [a, f(a)]$  (a *fundamental domain* in [10])

$$h_0 : [a, f(a)] \rightarrow [b, g(b)],$$

such that  $h_0(a) = b$  and  $h_0(f(a)) = g(b)$ <sup>2</sup>.

Note that we have chosen arbitrarily the homeomorphism  $h_0$  subject to the condition

$$g(h_0(a)) = h_0(f(a)).$$

We now use  $h_0$  to construct a topological conjugacy between  $g$  and  $f$ . The main ingredient of the proof is the non-locality of this construction process.

We have started the construction of a topological equivalence by the restriction of  $h$  to the fundamental domain which is  $h_0$ . Consider the points  $x$  in the fundamental domain  $D_0 = [a, f(a)]$ , the right side of the conjugacy equation (1), i.e.,  $h \circ f(x)$ , acts on  $D_0$ . Obviously  $f(D_0) = [f(a), f^2(a)] = D_1$ . In order to compute directly  $h(x)$  when  $x \in D_1$ , we use the left hand side of the conjugacy equation and define  $h_1(x)$  when  $x \in D_1$ , i.e., the restriction of  $h$  to the interval  $D_1$ . The left side of the conjugacy equation when  $x \in D_0$  is well defined, it is  $g \circ h_0(x)$ . We obtain a definition of  $h_1(x)$ , i.e., the restriction of  $h(x)$  to the interval  $D_1$ , forcing the diagram to be commutative, that is,

$$h_1 \circ f(x) = g \circ h_0(x), \quad x \in D_0$$

or

$$h_1(x) = (g \circ h_0 \circ f^{-1})(x), \quad x \in D_1.$$

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<sup>2</sup>For instance,

$$h_0(x) = b + (x - a) \frac{g(b) - b}{f(a) - a}, \quad x \in D_0$$

works perfectly to start the process. Any other continuous map with the same properties works just fine.

In such a way, we can define the sequence of intervals

$$D_n = [f^n(a), f^{n+1}(a)] = f^n(D_0),$$

where  $f^n$  stands for the  $n$ -th composition of  $f$  with itself, and it is possible to extend the definition of successive restrictions  $h_n$  of  $h$  to the intervals  $D_n$ , using the same procedure. Thus, for  $n = 2$ , we get

$$h_2(x) = (g \circ h_1 \circ f^{-1})(x), \quad x \in D_2$$

or

$$h_2(x) = (g^2 \circ h_0 \circ f^{-2})(x), \quad x \in D_2,$$

where  $f^{-2}$  stands for  $f^{-1} \circ f^{-1}$ . In general

$$h_n(x) = (g^n \circ h_0 \circ f^{-n})(x), \quad x \in D_n.$$

This construction is well done. Effectively, let  $x \in D_j$  with  $0 < j < n$ , then

$$\begin{aligned} g \circ h_j(x) &= h_{j+1} \circ f(x) \\ g \circ (g^j \circ h_0 \circ f^{-j})(x) &= (g^{j+1} \circ h_0 \circ f^{-j-1}) \circ f(x) \\ g^{j+1} \circ h_0 \circ f^{-j}(x) &= g^{j+1} \circ h_0 \circ f^{-j}(x). \end{aligned}$$

Consider now the image domains

$$D'_n = [g^n(b), g^{n+1}(b)]$$

and the homeomorphisms

$$h_n : D_n \longrightarrow D'_n, \quad h_n(x) = (g^n \circ h_0 \circ f^{-n})(x),$$

which are increasing in each of the domains, since they are the composition of increasing functions.

At this point, observe that the union of the intervals  $D_n = [f^n(a), f^{n+1}(a)]$  is

$$\bigcup_{n=0}^{\infty} D_n = [a, \bar{x}_F),$$

while the union of the intervals  $D'_n = [g^n(b), g^{n+1}(b)]$  is

$$\bigcup_{n=0}^{\infty} D'_n = [b, x_F),$$

This is true, since the sequence  $(f^n(a))_{n \in \mathbb{N}}$  is increasing and bounded by  $\bar{x}_F$ . Therefore, it has a limit, namely  $L$ . But, due to the continuity of the  $f$ ,

$$f(L) = f\left(\lim_{n \rightarrow \infty} f^n(a)\right) = \lim_{n \rightarrow \infty} f^{n+1}(a) = L$$

Hence, the limit of the sequence is a fixed point of  $f$  and, by hypothesis 4, the unique fixed point in the interval  $[a, \bar{x}_F]$  is precisely  $\bar{x}_F$ . A similar reasoning can be used for the case of  $g$  and the union of the intervals  $D'_n$ .

Then, we can consider the piecewise function  $h^\circ$  defined as

$$h^\circ(x) = \begin{cases} h_0(x) & : & [a, f(a)] & \rightarrow & [b, g(b)] \\ h_1(x) & : & [f(a), f^2(a)] & \rightarrow & [g(b), g^2(b)] \\ \dots & & & & \\ h_n(x) & : & [f^n(a), f^{n+1}(a)] & \rightarrow & [g^n(b), g^{n+1}(b)] \\ \dots & & & & \end{cases}$$

The function  $h^\circ$  is a (increasing) homeomorphism piecewise defined in  $[a, \bar{x}_F]$  which is constituted by increasing homeomorphisms joined together. Finally, we extend  $h^\circ$  by continuity to a new homeomorphism, our desired  $h$ , such that

$$\begin{aligned} h(x) &= h^\circ(x), \text{ for } x \in [a, \bar{x}_F] \\ h(\bar{x}_F) &= x_F. \end{aligned}$$

The homeomorphism  $h$  defined in  $[a, \bar{x}_F]$  is the required conjugation. ■

**Corollary 2.5** *Let  $g : I \rightarrow g(I)$  and  $f : J \rightarrow f(J)$  two increasing homeomorphisms with domains  $I = [x_F, b]$  and  $J = [\bar{x}_F, a]$ . Suppose that they verify the following conditions:*

1.  $g(x) < x$ , for all  $x \in (x_F, b]$ .
2.  $f(x) < x$ , for all  $x \in (\bar{x}_F, a]$ .
3.  $x_F$  is the (unique) fixed point of  $g$  and  $\bar{x}_F$  is the (unique) fixed point of  $f$ .

*Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : J \rightarrow I$ , not necessarily unique, such that*

$$g \circ h(x) = h \circ f(x).$$

**Proof.** We note that both  $x_F$  and  $\bar{x}_F$  are semi-attracting from the right. The proof is similar to the one of Theorem 2.4. The unique difference is that, in this case, due to the hypotheses 1 and 2, now the domains are

$$\begin{aligned} D_n &= [f^{n+1}(a), f^n(a)], \\ D'_n &= [g^{n+1}(b), g^n(b)] \end{aligned}$$

and the homeomorphisms are

$$h_n : D_n \longrightarrow D'_n, \quad h_n(x) = (g^n \circ h_0 \circ f^{-n})(x)$$

■

**Corollary 2.6** *Let  $g : I \longrightarrow g(I)$  and  $f : J \longrightarrow f(J)$  two increasing homeomorphisms with domains  $I = [b, x_F]$  and  $J = [a, \bar{x}_F]$ . Suppose that they verify the following conditions:*

1.  $g(x) < x$ , for all  $x \in [b, x_F]$ .
2.  $f(x) < x$ , for all  $x \in [a, \bar{x}_F]$ .
3.  $x_F$  is the (unique) fixed point of  $g$  and  $\bar{x}_F$  is the (unique) fixed point of  $f$ .

*Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : J \longrightarrow I$ , not necessarily unique, such that*

$$g \circ h(x) = h \circ f(x).$$

**Proof.** First of all, we note that both  $x_F$  and  $\bar{x}_F$  are both semi-repelling to the left. The proof is similar to the one of Theorem 2.4. Consider  $a$ , the leftmost point of  $J$ . Since  $f(x) < x$  we have  $f^{-1}(x) > x$ , so  $f^{-1}(a) > a$ . The same happens to  $g$ , so  $g^{-1}(b) > b$ . We note that  $f^{-1}$  and  $g^{-1}$  have the same fixed points of  $f$  and  $g$ , which are now attracting. The sequences  $f^{-n}(a)$  and  $g^{-n}(x)$  are increasing and bounded by the fixed points  $\bar{x}_F$  and  $x_F$ .

At this point, we can apply Theorem 2.4 to  $f^{-1}$  and  $g^{-1}$ . Therefore,  $f^{-1}$  and  $g^{-1}$  are topologically equivalent

$$g^{-1} = h \circ f^{-1} \circ h^{-1},$$

with  $h : J \longrightarrow I$ . Thus inverting the maps in the last equality, we get

$$g = h \circ f \circ h^{-1},$$

as desired. ■

**Corollary 2.7** *Let  $g : I \longrightarrow g(I)$  and  $f : J \longrightarrow f(J)$  two increasing homeomorphisms with domains  $I = [x_F, b]$  and  $J = [\bar{x}_F, a]$ . Suppose that they verify the following conditions*

1.  $g(x) > x$ , for all  $x \in (x_F, b]$ .
2.  $f(x) > x$ , for all  $x \in (\bar{x}_F, a]$ .
3.  $x_F$  is the (unique) fixed point of  $g$  and  $\bar{x}_F$  is the (unique) fixed point of  $f$ .



Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : J \longrightarrow I$ , not necessarily unique, such that

$$g \circ h(x) = h \circ f(x).$$

**Proof.** First of all we note that both  $x_F$  and  $\bar{x}_F$  are semi-repelling to the right. The proof is similar to the ones in the previous results. ■

## 2.2 Maps with consecutive pairs of fixed points

Consider two homeomorphisms  $f$  and  $g$  with a finite number of isolated fixed points. In this subsection, we study the construction of a homeomorphism  $h$  in intervals between two fixed points of  $g$  and  $f$ . Of course, the obtained results can be applied between any two consecutive pairs of fixed points of  $g$  and  $f$ . The exterior of the union of such intervals between two consecutive fixed points can be treated using the results of the previous subsection.

**Theorem 2.8** *Let  $g : I \longrightarrow I$  and  $f : J \longrightarrow J$  two increasing homeomorphisms defined in the intervals  $I = [x_L, x_R]$  and  $J = [\bar{x}_L, \bar{x}_R]$ . Suppose that they verify the following conditions:*

1.  $g(x) > x$ , for all  $x \in I = (x_L, x_R)$ .
2.  $f(x) > x$ , for all  $x \in J = (\bar{x}_L, \bar{x}_R)$ .
3. The endpoints of the intervals  $I, J$  are the unique fixed points of  $g$  and  $f$  respectively.

Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : J \longrightarrow I$ , not necessarily unique, such that

$$g \circ h(x) = h \circ f(x) \tag{2}$$

**Proof.** First of all, it is easy to check that  $x_L$  and  $\bar{x}_L$  are semi-repelling fixed points to the right while  $x_R$  and  $\bar{x}_R$  are semi-attracting fixed points from the left.

To prove the result, we fix one image of  $h$  at one particular point

$$h(a) = b,$$

where  $\bar{x}_L < a < \bar{x}_R$  and  $x_L < b < x_R$ .

Now, we can consider an increasing homeomorphism  $h_0$  in the domain  $D_0 = [a, f(a)]$

$$h_0 : [a, f(a)] \longrightarrow [b, g(b)],$$

such that  $h_0(a) = b$  and  $h_0(f(a)) = g(b)$ . Note that we have chosen arbitrarily the homeomorphism  $h_0$  subject to the condition  $g(h_0(a)) = h_0(f(a))$ .

We use  $h_0$  to construct a topological conjugacy between  $g$  and  $f$ . The procedure is divided in two steps: in the first one, we construct the conjugacy at forward intervals, i.e., right of the fundamental domain; in the second one, we construct the conjugacy at backward intervals, i.e., left of the fundamental domain.

We have started the construction of a topological equivalence by the restriction of  $h$  to the fundamental domain which is  $h_0$ . Consider the points  $x$  in the fundamental domain  $D_0 = [a, f(a)]$ , the right side of the conjugacy equation (2), i.e.,  $h \circ f(x)$ , acts on  $D_0$ . Obviously  $f(D_0) = [f(a), f^2(a)] = D_1$ . In order to compute directly  $h(x)$  when  $x \in D_1$ , we use the left hand side of the conjugacy equation and define  $h_1(x)$  when  $x \in D_1$ , i.e., the restriction of  $h$  to the interval  $D_1$ . The left side of the conjugacy equation when  $x \in D_0$  is well defined, it is  $g \circ h_0(x)$ . We obtain a definition of  $h_1(x)$ , i.e., the restriction of  $h(x)$  to the interval  $D_1$ , forcing the diagram to be commutative, that is,

$$h_1 \circ f(x) = g \circ h_0(x), x \in D_0$$

or

$$h_1(x) = (g \circ h_0 \circ f^{-1})(x), x \in D_1.$$

In such a way, we can define the sequence of intervals

$$D_n = [f^n(a), f^{n+1}(a)] = f^n(D_0),$$

where  $f^n$  stands for the  $n$ -th composition of  $f$  with itself, and it is possible to extend the definition of successive restrictions  $h_n$  of  $h$  to the intervals  $D_n$ , using the same procedure. In general

$$h_n(x) = (g^n \circ h_0 \circ f^{-n})(x), x \in D_n.$$

At this point, observe that the union of the intervals  $D_n = [f^n(a), f^{n+1}(a)]$  is

$$\bigcup_{n=0}^{\infty} D_n = [a, \bar{x}_R],$$

This is true, since the sequence  $(f^n(a))_{n \in \mathbb{N}}$  is increasing and bounded by  $\bar{x}_R$ . Therefore, it has a limit, namely  $L$ . But, due to the continuity of the  $f$ ,

$$f(L) = f\left(\lim_{n \rightarrow \infty} f^n(a)\right) = \lim_{n \rightarrow \infty} f^{n+1}(a) = L$$

Hence, the limit of the sequence is a fixed point of  $f$  and, by hypothesis 4, the unique fixed point in the interval  $[a, \bar{x}_R]$  is precisely  $\bar{x}_R$ .

Summarizing, we have the function  $h_{\rightarrow}^{\circ}$  piecewise defined as

$$h_{\rightarrow}^{\circ}(x) = \begin{cases} h_0(x) & : & [a, f(a)] & \rightarrow & [b, g(b)] \\ h_1(x) & : & [f(a), f^2(a)] & \rightarrow & [g(b), g^2(b)] \\ \dots & & & & \\ h_n(x) & : & [f^n(a), f^{n+1}(a)] & \rightarrow & [g^n(b), g^{n+1}(b)] \\ \dots & & & & \end{cases}$$

Of course,  $h_{\rightarrow}^{\circ}$  is continuous and increasing in each interval and agrees at each common extreme point of two successive intervals. Thus,  $h_{\rightarrow}^{\circ}$  is a homeomorphism. Finally, observe that, as in the case of  $f$ , the sequence  $(g^n(a))_{n \in \mathbb{N}}$  is increasing and bounded by  $x_R$ . Therefore, it has a limit and due to the continuity of the  $g$  and the hypothesis 4, its limit is precisely  $x_R$ . As a consequence, we can prolong  $h_{\rightarrow}^{\circ}$  to  $\bar{x}_R$  by continuity defining the complete homeomorphism  $h_{\rightarrow}$  in the compact interval  $[a, \bar{x}_R]$  making  $h_{\rightarrow}(\bar{x}_R) = x_R$  and  $h_{\rightarrow}(x) = h_{\rightarrow}^{\circ}(x)$  for  $x \in [a, \bar{x}_R)$ , i.e.,

$$h_{\rightarrow} : [a, \bar{x}_R] \longrightarrow [a, x_R],$$

Using a similar reasoning, we can construct the topological conjugacy to the left of the fundamental domain using now  $f^{-1}$  and  $g^{-1}$ .

The restriction of  $h$  to the fundamental domain is again  $h_0$ . Consider the points  $x$  in the fundamental domain  $D_0 = [a, f(a)]$ . Now, we need to define the restrictions  $h_{-n}$  of  $h$  to the intervals  $D_{-n} = [f^{-n}(a), f^{-n+1}(a)]$ . Using the same type of arguments of the forward construction we find that

$$h_{-n}(x) = g^{-n} \circ h_0 \circ f^n(x), \quad x \in D_{-n}.$$

Effectively, let  $x \in D_{-j}$  with  $-n < -j \leq 0$ , then

$$\begin{aligned} g \circ h_{-j}(x) &= h_{-j+1} \circ f(x) \\ g \circ (g^{-j} \circ h_0 \circ f^j)(x) &= (g^{-j+1} \circ h_0 \circ f^{j-1}) \circ f(x) \\ g^{-j+1} \circ h_0 \circ f^j(x) &= g^{-j+1} \circ h_0 \circ f^j(x). \end{aligned}$$

The rest of the backward process is just the same as in the forward case, since now  $\bar{x}_L$  and  $x_L$  are attracting fixed points for  $f^{-1}$  and  $g^{-1}$ .

Summarizing again, we can define the piecewise homeomorphism  $h_{\leftarrow}^{\circ}$

$$h_{\leftarrow}^{\circ}(x) = \begin{cases} h_{-1}(x) & : & [f^{-1}(a), a] & \rightarrow & [g^{-1}(b), b] \\ h_{-2}(x) & : & [f^{-2}(a), f^{-1}(a)] & \rightarrow & [g^{-2}(b), g^{-1}(b)] \\ \dots & & & & \\ h_{-n}(x) & : & [f^{-n}(a), f^{-n+1}(a)] & \rightarrow & [g^{-n}(b), g^{-n+1}(b)] \\ \dots & & & & \end{cases}.$$

The homeomorphism  $h_{\leftarrow}^{\circ}(x)$  defined in  $(\bar{x}_L, a]$  can be prolonged by continuity to  $\bar{x}_L$  the same way as  $h_{\rightarrow}$ .

Therefore, joining the backward homeomorphism  $h_{\leftarrow}$  and the forward one  $h_{\rightarrow}$ , we obtain the increasing homeomorphism  $h$  at the whole interval  $J = [\bar{x}_L, \bar{x}_R]$  and, as desired, the image of  $h$  is  $[x_L, x_R]$ . ■

**Remark 2.9** *Actually, Theorem 2.8 can be seen as a corollary of Theorem 2.4 applied directly to the right fixed points (in the intervals  $[a, \bar{x}_R]$  for  $f$  and  $[b, x_R]$  for  $g$ ) and indirectly, using Corollary 2.7 applied to the left fixed points (in the intervals  $[\bar{x}_F, a]$  for  $f$  and  $[x_F, b]$  for  $g$ ).*

Naturally, a similar reasoning gives the same result when both semi-attracting fixed points are on the left and the semi-repelling fixed points on the right. Specifically, one can check the following corollary.

**Corollary 2.10** *Let  $g : I \rightarrow I$  and  $f : J \rightarrow J$  two increasing homeomorphisms defined in the intervals  $I = [x_L, x_R]$  and  $J = [\bar{x}_L, \bar{x}_R]$ . Suppose that they verify the following conditions:*

1.  $g(x) < x$ , for all  $x \in I = (x_L, x_R)$ .
2.  $f(x) < x$ , for all  $x \in J = (\bar{x}_L, \bar{x}_R)$ .
3. The endpoints of the intervals  $I, J$  are the unique fixed points of  $g$  and  $f$  respectively.

*Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : J \rightarrow I$ , not necessarily unique, such that*

$$g \circ h(x) = h \circ f(x). \quad (3)$$

**Proof.** Although in this case  $x_L$  and  $\bar{x}_L$  are semi-attracting fixed points from the right while  $x_R$  and  $\bar{x}_R$  are semi-repelling fixed points to the left, the proof is similar to the one in Theorem 2.8. ■

Actually, the position of semi-attracting and semi-repelling fixed points is not relevant. The semi-attracting and semi-repelling fixed points can be opposite

for  $g$  or  $f$ . The original map  $g$  is still conjugated to  $f$ , thanks to a reverse order homeomorphism.

**Corollary 2.11** *Let  $g : I \rightarrow I$  and  $f : J \rightarrow J$  two increasing homeomorphisms defined in the intervals  $I = [x_L, x_R]$  and  $J = [\bar{x}_L, \bar{x}_R]$ . Suppose that they verify the following conditions:*

1.  $g(x) < x$ , for all  $x \in I = (x_L, x_R)$ .
2.  $f(x) > x$ , for all  $x \in J = (\bar{x}_L, \bar{x}_R)$ .
3. The endpoints of the intervals  $I, J$  are the unique fixed points of  $g$  and  $f$  respectively.

*Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : J \rightarrow I$ , not necessarily unique, such that*

$$g \circ h(x) = h \circ f(x). \quad (4)$$

**Proof.** First of all note that, in this case,  $x_L$  and  $\bar{x}_R$  are semi-attracting fixed points while  $x_R$  and  $\bar{x}_L$  are semi-repelling fixed points.

The proof is quite similar to the one in Theorem 2.8. The unique difference is that, in this case, as  $g(x) < x$  for all  $x \in I = (x_L, x_R)$ , we define a decreasing homeomorphism  $h_0$  in the domain  $D_0 = [a, f(a)]$

$$h_0 : [a, f(a)] \rightarrow [g(b), b],$$

such that  $h_0(a) = b$  and  $h_0(f(a)) = g(b)$ .

In such a context, the new domains are

$$D_n = [f^n(a), f^{n+1}(a)],$$

$$D'_n = [g^{n+1}(b), g^n(b)]$$

and the homeomorphisms are

$$h_n : D_n \rightarrow D'_n, \quad h_n(x) = (g^n \circ h_0 \circ f^{-n})(x),$$

which are decreasing in each of the domains.

The same change has to be considered for the domains

$$D_{-n} = [f^{-(n+1)}(a), f^{-n}(a)],$$

$$D'_{-n} = [g^{-n}(b), g^{-(n+1)}(b)]$$

and the homeomorphisms

$$h_{-n} : D_{-n} \rightarrow D'_{-n}, \quad h_{-n}(x) = (g^{-n} \circ h_0 \circ f^n)(x),$$

With these considerations, similar arguments to those in Theorem 2.8 work.

■

**Remark 2.12** *As a consequence of the above Corollary 2.11, it can be deduced that any increasing homeomorphism  $f : J \rightarrow J$  with two fixed points,  $\bar{x}_L, \bar{x}_R$  which are the endpoints of the interval  $J$ , is topologically equivalent to its inverse  $f^{-1} : J \rightarrow J$ . Furthermore, in such a case,  $h(x) = \bar{x}_L + \bar{x}_R - x$  is a topological conjugacy between them.*

### 2.3 Maps with $n$ fixed points

The combination of all the results of the previous subsections leaves us in position to state the following more general result about increasing homeomorphisms.

**Theorem 2.13** *Let  $g : I \rightarrow g(I)$  and  $f : J \rightarrow f(J)$  two increasing homeomorphisms. Then, they are topologically conjugated if and only if they have the same number of fixed points with the same sequence of stabilities or reversed sequence of stabilities.*

**Proof.** The proof results immediately from a combination and repeated application to increasing sets of Theorem 2.4 and Corollaries 2.5, 2.6 and 2.7 for one fixed point (or intervals lying in the exterior of the leftmost and rightmost fixed points) and Theorem 2.8 and Corollaries 2.10 and 2.11 for intervals between pairs of fixed points and observing that two homeomorphisms with zero fixed points are trivially conjugated.

If the order of the stabilities of the fixed points is reversed from one map to the other, we use reverse homeomorphisms constructed using a reasoning similar to Remark 2.12 to obtain the conjugacy. ■

**Example 2.14** *Consider three homeomorphisms  $\phi$ ,  $\gamma$  and  $\zeta$  with three fixed points each one,  $\phi$  and  $\gamma$  have the leftmost fixed point attracting, the middle one semi-repelling to the left and semi-attracting from the right and the rightmost fixed point repelling, finally  $\zeta$  has the leftmost fixed point repelling, the middle one semi-attracting from the left and semi-repelling to the right and with the rightmost fixed point attracting. Accordingly to Theorem 2.13 the three homeomorphisms are topologically equivalent.*

The general Theorem 2.13 is enough to prove all the results of topological conjugacy for bifurcations with eigenvalue 1 in the section 3.

**Remark 2.15** *As a consequence of Theorem 2.13, two increasing homeomorphisms with the same even number of transverse fixed points are topologically conjugated.*

**Remark 2.16** *Observe that the Theorem 2.13, allows us to deduce that any increasing homeomorphism with an even number of transverse fixed points is topologically conjugated to its inverse. Moreover, we can affirm that any increasing homeomorphism with an odd number of transverse fixed points is not topologically conjugated to its inverse.*

## 2.4 Maps with one fixed point and one 2-periodic orbit

In this section, we deal with conjugacies between decreasing homeomorphisms associated to the flip bifurcation. For simplicity, we shall consider that  $g$  and  $f$  have a fixed point<sup>3</sup> at the origin. We consider first the topological conjugacy of two homeomorphisms  $g$  and  $f$  between the two periodic points of each homeomorphism.

**Theorem 2.17** *Let  $g : I \rightarrow I$  and  $f : J \rightarrow J$  be decreasing homeomorphisms with domains  $I = [x_L, x_R]$  and  $J = [\bar{x}_L, \bar{x}_R]$ . Suppose that they verify the following conditions:*

1.  $g(x)$  has the repelling fixed point 0 and an attracting period two orbit  $\{x_L, x_R\}$ , with  $x_L < 0$  and  $x_R > 0$ , such that  $g(x_L) = x_R$  and  $g(x_R) = x_L$ .
2.  $f(x)$  has the repelling fixed point 0 and an attracting period two orbit  $\{\bar{x}_L, \bar{x}_R\}$ , with  $\bar{x}_L < 0$  and  $\bar{x}_R > 0$ , such that  $f(\bar{x}_L) = \bar{x}_R$  and  $f(\bar{x}_R) = \bar{x}_L$ .
3.  $g^2(x) > x$ , for all  $x \in (0, x_R)$ .
4.  $f^2(x) > x$ , for all  $x \in (0, \bar{x}_R)$ .

Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : [\bar{x}_L, \bar{x}_R] \rightarrow [x_L, x_R]$ , not necessarily unique, such that

$$g \circ h(x) = h \circ f(x). \quad (5)$$

**Proof.** To prove the result, we fix one image of  $h$  at one particular point

$$h(a) = b,$$

where  $0 < a < \bar{x}_R$  and  $0 < b < x_R$ . We will construct the homeomorphism  $h$  subject to the condition  $h(a) = b$ .

Now, we consider an arbitrary increasing homeomorphism  $h_0$  from the domain  $D_0 = [a, f^2(a)]$  onto  $D'_0 = [b, g^2(b)]$

$$h_0 : [a, f^2(a)] \rightarrow [b, g^2(b)],$$

subject to the conditions  $h_0(a) = b$  and  $h_0(f^2(a)) = g^2(b)$ . This homeomorphism is increasing due to hypotheses 3 and 4.

This construction allows us to define the homeomorphism  $h_1(x)$  in the domain

$$D_1 = [f^3(a), f(a)] \rightarrow D'_1 = [g(b), g^3(b)]$$

and to construct step by step the complete topological conjugacy between  $g$  and  $f$ .

---

<sup>3</sup>In the case of decreasing homeomorphisms, if there exists a unique fixed point, it must be transverse.

Then, the restriction of  $h$  to  $D_0$  is  $h_0$  and to  $D_1$  is  $h_1$ . Consider the points  $x$  in the domain  $D_0 = [a, f^2(a)]$ , the right side of the conjugacy equation, i.e.,  $h \circ f(x)$ , acts on  $D_0$ , obviously  $f(D_0) = [f^3(a), f(a)] = D_1$ .

In order to compute  $h(x)$  when  $x \in D_1$ , we use the left hand side of the conjugacy equation (5) to define  $h_1(x)$  when  $x \in D_1$ , i.e., the restriction of  $h$  to the interval  $D_1$  using the rule

$$h_1(x) = (g \circ h_0 \circ f^{-1})(x), x \in D_1.$$

Once again, this homeomorphism is increasing due to the fact that  $f^{-1}$  and  $g$  are decreasing and  $h_0$  is increasing.

Defining the sequence of intervals

$$\begin{aligned} D_{2n} &= [f^{2n}(a), f^{2n+2}(a)] = f^{2n}(D_0), \\ D_{2n+1} &= [f^{2n+3}(a), f^{2n+1}(a)] = f^{2n+1}(D_0), \end{aligned}$$

it is possible to extend the definition of successive restrictions of  $h$  to the intervals  $D_n$ , using the same procedure. Thus, for  $n = 2$ , we get

$$h_2(x) = (g \circ h_1 \circ f^{-1})(x), x \in D_2$$

or

$$h_2(x) = (g^2 \circ h_0 \circ f^{-2})(x), x \in D_2.$$

In general

$$h_n(x) = (g^n \circ h_0 \circ f^{-n})(x), x \in D_n,$$

which is increasing.

Let us see that this definition works. Let  $x \in D_j$ ,

$$\begin{aligned} g \circ h_j(x) &= h_{j+1} \circ f(x) \\ g \circ (g^j \circ h_0 \circ f^{-j})(x) &= (g^{j+1} \circ h_0 \circ f^{-j-1}) \circ f(x) \\ g^{j+1} \circ h_0 \circ f^{-j}(x) &= g^{j+1} \circ h_0 \circ f^{-j}(x). \end{aligned}$$

Following a similar reasoning as in theorems before, one can easily check that the union of the intervals  $D_{2n} = [f^{2n}(a), f^{2n+2}(a)]$  is

$$\bigcup_{n=0}^{\infty} D_{2n} = [a, \bar{x}_R],$$

because  $\bar{x}_R$  is the unique positive fixed point of  $f^2$  and the initial condition  $a$  is positive. Analogously, the union of the intervals  $D_{2n+1} = [f^{2n+3}(a), f^{2n+1}(a)]$  is

$$\bigcup_{n=0}^{\infty} D_{2n+1} = (\bar{x}_L, f(a)],$$

since  $\bar{x}_L$  is an attracting fixed point of  $f^2$  and the initial condition  $f(a)$  is negative.



On the other hand, the piecewise function  $h_{\rightarrow}^{\circ}$  defined by

$$h_{\rightarrow}^{\circ}(x) = \left\{ \begin{array}{llll} \dots & & & \\ h_{2n+1}(x) : & [f^{2n+3}(a), f^{2n+1}(a)] & \rightarrow & [g^{2n+3}(b), g^{2n+1}(b)] \\ \dots & & & \\ h_3(x) : & [f^5(a), f^3(a)] & \rightarrow & [g^5(b), g^3(b)] \\ h_1(x) : & [f^3(a), f(a)] & \rightarrow & [g^3(b), g(b)] \\ h_0(x) : & [a, f^2(a)] & \rightarrow & [b, g^2(b)] \\ h_2(x) : & [f^2(a), f^4(a)] & \rightarrow & [g^2(b), g^4(b)] \\ \dots & & & \\ h_{2n}(x) : & [f^{2n}(a), f^{2n+2}(a)] & \rightarrow & [g^{2n}(b), g^{2n+2}(b)] \\ \dots & & & \end{array} \right.$$

is continuous in each interval and agrees at each extreme point of two successive intervals. Moreover,  $h_j = g^j \circ h_0 \circ f^{-j}$  is the composition of two decreasing functions with an increasing function when  $j$  is odd and three increasing functions when  $j$  is even. Thus, it is increasing in  $D_j$ . We can prolong again  $h_{\rightarrow}^{\circ}$  by continuity to  $h_{\rightarrow}$  in the closed interval making  $h(\bar{x}_L) = x_L$  and  $h(\bar{x}_R) = x_R$ . This implies that  $h_{\rightarrow}$  is an increasing homeomorphism in  $[\bar{x}_L, f(a)]$  and in  $[a, \bar{x}_R]$ .

The second part of the proof concerns the backward process. Using a similar reasoning, we construct the topological conjugacy to the pre-images of  $D_0$ .

We remember that the restriction of  $h$  to  $D_0$  is  $h_0$ . Consider the points  $x$  in  $D_0 = [a, f^2(a)]$ , then we can define the restrictions  $h_{-n}$  of  $h$  to the intervals

$$\begin{aligned} D_{-2n} &= [f^{-2n}(a), f^{-2n+2}(a)] = f^{-2n}(D_0), \\ D_{-2n+1} &= [f^{-2n+3}(a), f^{-2n+1}(a)] = f^{-2n+1}(D_0), \end{aligned}$$

for  $n \geq 1$ .

Using the same type of arguments of the forward construction, we find that

$$h_{-n}(x) = g^{-n} \circ h_0 \circ f^n(x), \quad x \in D_{-n}.$$

As in Theorem 2.8 this type of construction works. Let  $x \in D_{-j}$  with  $-n < -j \leq 0$ , then

$$\begin{aligned} g \circ h_{-j}(x) &= h_{-j+1} \circ f(x) \\ g \circ (g^{-j} \circ h_0 \circ f^j)(x) &= (g^{-j+1} \circ h_0 \circ f^{j-1}) \circ f(x) \\ g^{-j+1} \circ h_0 \circ f^j(x) &= g^{-j+1} \circ h_0 \circ f^j(x). \end{aligned}$$

The rest of the backward process is just the same as in the forward case, since now 0 is an attracting fixed point for both  $f^{-1}$  and  $g^{-1}$ . We define  $h_{\leftarrow}^{\circ}$  piecewise

$$h_{\leftarrow}^{\circ}(x) = \begin{cases} h_{-1}(x) : & [f(a), f^{-1}(a)] \rightarrow [g(b), g^{-1}(b)] \\ h_{-3}(x) : & [f^{-1}(a), f^{-3}(a)] \rightarrow [g^{-1}(b), g^{-3}(b)] \\ \dots & \\ h_{-2n+1}(x) : & [f^{-2n+3}(a), f^{-2n+1}(a)] \rightarrow [g^{-2n+3}(b), g^{-2n+1}(b)] \\ \dots & \\ h_{-2n}(x) : & [f^{-2n}(a), f^{-2n+2}(a)] \rightarrow [g^{-2n}(b), g^{-2n+2}(b)] \\ \dots & \\ h_{-4}(x) : & [f^{-4}(a), f^{-2}(a)] \rightarrow [g^{-4}(b), g^{-2}(b)] \\ h_{-2}(x) : & [f^{-2}(a), a] \rightarrow [g^{-2}(b), b] \end{cases}$$

Noticing that  $\lim_{n \rightarrow \infty} f^{-n}(a) = 0$  and  $\lim_{n \rightarrow \infty} g^{-n}(b) = 0$ , we prolong  $h_{\leftarrow}^{\circ}$  to a homeomorphism  $h_{\leftarrow}$  making  $h_{\leftarrow}(0) = 0$  as before.

Now, joining  $h_{\leftarrow}$  and  $h_{\rightarrow}$  at their corresponding intervals, we have just constructed  $h$  at the whole interval  $J = [\bar{x}_L, \bar{x}_R]$ , as desired. ■

Now, we have to consider what happens outside the intervals that contain the periodic points of both  $g$  and  $f$ .

**Theorem 2.18** *Let  $g : I \rightarrow I$  and  $f : J \rightarrow J$  be decreasing homeomorphisms with domains  $I = [g(b), b]$  and  $J = [f(a), a]$ . Suppose that they verify the conditions:*

1.  $g(x)$  has the repelling fixed point 0 and an attracting period two orbit  $\{x_L, x_R\}$  with  $x_L < 0$  and  $x_R > 0$ , such that  $g(x_L) = x_R$  and  $g(x_R) = x_L$ . Moreover  $b > x_R$ .
2.  $f(x)$  has the repelling fixed point 0 and an attracting period two orbit  $\{\bar{x}_L, \bar{x}_R\}$  with  $\bar{x}_L < 0$  and  $\bar{x}_R > 0$ , such that  $f(\bar{x}_L) = \bar{x}_R$  and  $f(\bar{x}_R) = \bar{x}_L$ . Moreover  $a > \bar{x}_R$ .
3.  $g^2(x) < x$ , for all  $x \in (x_R, b]$ .
4.  $f^2(x) < x$ , for all  $x \in (\bar{x}_R, a]$ .

Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism

$$h : [f(a), \bar{x}_L] \cup [\bar{x}_R, a] \rightarrow [g(b), x_L] \cup [x_R, b]$$

not necessarily unique, such that

$$g \circ h(x) = h \circ f(x),$$

**Proof.** The proof is similar to the proof of Theorem 2.4. Consider  $a$ , the rightmost point of  $[\bar{x}_R, a]$ . Since  $f^2(x) < x$ , we have  $f^2(a) < a$ . Consider the interval  $D_0 = [f^2(a), a]$  and any homeomorphism  $h_0(x)$  subject to the conditions  $h_0(a) = b$  and  $h_0(f^2(a)) = g^2(b)$ . Due to conditions 3 and 4 above,  $h_0$  is increasing since  $f^2(a) < a$  and  $g^2(b) < b$ .

Consider now domains

$$\begin{aligned} D_{2n} &= [f^{2n+2}(a), f^{2n}(a)], \\ D_{2n+1} &= [f^{2n+1}(a), f^{2n+3}(a)] \end{aligned}$$

and

$$\begin{aligned} D'_{2n} &= [g^{2n+2}(b), g^{2n}(b)] \\ D'_{2n+1} &= [g^{2n+1}(b), g^{2n+3}(b)] \end{aligned}$$

and the homeomorphisms

$$h_n : D_n \longrightarrow D'_n, \quad h_n(x) = (g^n \circ h_0 \circ f^{-n})(x),$$

which are increasing in each domain, since they are the composition of increasing functions (when  $n$  is even) or two decreasing functions and an increasing function (when  $n$  is odd).

At this point, we can consider the piecewise function  $h^\circ$  defined by

$$h^\circ(x) = \begin{cases} h_1(x) : & [f(a), f^3(a)] \rightarrow [g(b), g^3(b)] \\ h_3(x) : & [f^3(a), f^5(a)] \rightarrow [g^3(b), g^5(b)] \\ \dots & \\ h_{2n+1}(x) : & [f^{2n+1}(a), f^{2n+3}(a)] \rightarrow [g^{2n+1}(b), g^{2n+3}(b)] \\ \dots & \\ h_{2n}(x) : & [f^{2n+2}(a), f^{2n}(a)] \rightarrow [g^{2n+2}(b), g^{2n}(b)] \\ \dots & \\ h_2(x) : & [f^4(a), f^2(a)] \rightarrow [g^4(b), g^2(b)] \\ h_0(x) : & [f^2(a), a] \rightarrow [g^2(b), b], \end{cases}$$

which is continuous. Since

$$\lim_{n \rightarrow \infty} f^{2n}(a) = \bar{x}_R \text{ and } \lim_{n \rightarrow \infty} g^{2n}(b) = x_R,$$

we have

$$\lim_{x \rightarrow \bar{x}_R} h^\circ(x) = x_R,$$

and since

$$\lim_{n \rightarrow \infty} f^{2n+1}(a) = \bar{x}_L \text{ and } \lim_{n \rightarrow \infty} g^{2n+1}(b) = x_L,$$

we have

$$\lim_{x \rightarrow \bar{x}_L} h^\circ(x) = x_L.$$

Prolonging  $h^\circ$  to  $h$  at  $\bar{x}_L$  and  $\bar{x}_R$  as before, we get the topological conjugacy with the desired properties. ■

Now, we are going to consider the case previous to the appearance of the period two orbit in the bifurcation, that is, with both periodic points collapsed to the origin.

**Theorem 2.19** *Let  $g : I \rightarrow I$  and  $f : J \rightarrow J$  be decreasing homeomorphisms with domains  $I = [f(a), a]$  and  $J = [g(b), b]$ . Suppose that they verify the conditions:*

1.  $f(x)$  and  $g(x)$  have the attracting fixed point 0.
2. Consider  $b > 0$ , we have  $g^2(x) < x$ , for all  $x \in (0, b]$ .
3. Consider  $a > 0$ ,  $f^2(x) < x$ , for all  $x \in (0, a]$ .

*Then, there exists a topological conjugacy between  $g$  and  $f$ , i.e., a homeomorphism  $h : [f(a), a] \rightarrow [g(b), b]$ , not necessarily unique, such that*

$$g \circ h(x) = h \circ f(x).$$

**Proof.** The proof is similar to the one of Theorem 2.18, but with  $\bar{x}_L$  and  $\bar{x}_R$  collapsed to the origin. ■

**Remark 2.20** *Similar results to Theorems 2.17, 2.18 and 2.19 can be obtained for decreasing homeomorphisms  $g(x)$  and  $f(x)$  that have an attracting fixed point at 0 and a repelling period two orbit or only a repelling fixed point.*

### 3 Solution for the conjugacy problem for the fold, transcritical, pitchfork and flip bifurcations normal forms

The results in the previous sections allows us to establish local topological conjugacies between any two homeomorphisms with the same number of fixed points (or period-2 points) and with the same stability. These more general results can be used, in particular, to give a complete proof for the problem of finding a homeomorphism providing the topological equivalence between any family verifying specific bifurcation conditions and the corresponding simplest (truncated) normal form, which remained unpublished until now, as said in [9].

Actually, they can be used in a more general context of appearance of these bifurcations under generalized conditions (see [3]), giving topological equivalences between any family satisfying specific bifurcation conditions of any order and the simplest (truncated) normal form. In particular, normal forms of any order for the same bifurcation (see [4]) are topologically equivalent. Thus, theorems below give a solution for this classical problem in bifurcation theory even in a more general way than originally posed three decades ago in [1].

Specifically, here we show how to construct homeomorphisms which allows us to find (local) topological equivalences among one-parameter families undergoing the same kind of bifurcation.

We have a local topological equivalence of families  $f_\alpha$  and  $g_\beta$  depending on parameters  $\alpha$  and  $\beta$  whenever [2, 9]:

1. there exists a homeomorphism of the parameter space  $p : \mathbb{R} \longrightarrow \mathbb{R}$ ,  $\beta = p(\alpha)$ ;
2. there is a parameter-dependent homeomorphism of the phase space  $h_\alpha : J \longrightarrow I$ ,  $y = h_\alpha(x)$ , mapping orbits of the system  $f_\alpha$  onto orbits of  $g_\beta$ , i.e.,  $h_\alpha \circ f_\alpha = g_\beta \circ h_\beta$ , at parameter values  $\beta = p(\alpha)$ .

As can be done, we consider the same denomination for the parameter  $\mu$  for the two maps  $f_\mu$  and  $g_\mu$  which we want to conjugate. Thus, the homeomorphism  $p$  is the identity,  $\alpha = \beta = \mu$  and  $h_\mu \circ f_\mu = g_\mu \circ h_\mu$ . One does not require the homeomorphism  $h_\mu$  to depend continuously on the parameter  $\mu$ , meaning that the conjugacy of the families  $f_\mu$  and  $g_\mu$  is a weak (or fiber) equivalence (see [2] or [9]).

**Theorem 3.1** (*Fold Normal Form*) *Let  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a one-parameter family of maps verifying the generalized nondegeneracy conditions of a fold bifurcation [3]. Then,  $f_\mu$  is (locally) topologically equivalent near the origin to one of the following normal forms*

$$g(x, \mu) = x \pm \mu \pm x^2 \tag{6}$$

**Proof.** It is sufficient to observe that on one side of  $\mu = 0$ , any element of the family  $f_\mu$  has two transverse fixed points, one unstable and one stable and  $g_\mu$

has exactly the same number of transverse fixed points one stable and the other unstable, therefore  $f_\mu$  and  $g_\mu$  satisfy the conditions of Theorem 2.13; while in the other side of  $\mu = 0$  any system  $f_\mu$  has no fixed points also satisfying Theorem 2.13.

On the other hand, the maps  $f_0$  and  $g_0$  have a unique mixed-stability fixed point with the same stability, i.e., semi-attracting from one side and semi-repelling to the other side or vice-versa. Consequently, both maps verify again Theorem 2.13 at the bifurcation point. ■

**Example 3.2** *The Theorem 3.1 allows us to know that every family of any of the forms*

$$x \pm \mu \pm x^2 + h.o.t.$$

*is locally topologically equivalent to one of the forms in (6), so giving a complete proof for the classical bifurcation problem exactly as posed in [1] or [9]. Moreover, it also allows us to know that any higher order normal form of the fold bifurcation*

$$x \pm \mu \pm x^{2n}, \quad n \in \mathbb{N}$$

*is topologically equivalent to the simplest one given in (6) and, even more generally, that any family of the form*

$$x \pm \mu \pm x^{2n} + h.o.t., \quad n \in \mathbb{N}$$

*is also topologically equivalent to one the simplest ones given in (6).*

**Theorem 3.3** *(Transcritical Normal Form) Let  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a one-parameter family of maps verifying the generalized nondegeneracy conditions of a transcritical bifurcation [3]. Then,  $f_\mu$  is (locally) topologically equivalent near the origin to one of the following normal forms*

$$g(x, \mu) = x \pm \mu x \pm x^2 \tag{7}$$

**Proof.** It is sufficient to observe that on both sides of  $\mu = 0$ , any system  $f_\mu$  has two transverse fixed points, which alternate their stabilities and  $g_\mu$  has exactly the same number of fixed points one stable and the other unstable. Hence, when  $\mu \neq 0$ ,  $f_\mu$  and  $g_\mu$  satisfy the conditions of Theorem 2.13, being topologically conjugated.

On the other hand, for  $f_0$  and  $g_0$  there exists for each map a unique mixed-stability fixed point. So, both maps verify again Theorem 2.13. ■

**Example 3.4** *The Theorem 3.3 allows us to know that every family of any the forms*

$$x \pm \mu x \pm x^{2n} + h.o.t., \quad n \in \mathbb{N}$$

*is locally topologically equivalent to one of the forms in (7).*

**Theorem 3.5** (*Pitchfork Normal Form*) Let  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a one-parameter family of maps verifying the generalized nondegeneracy conditions of a pitchfork bifurcation [3]. Then,  $f_\mu$  is (locally) topologically equivalent near the origin to one of the following normal forms

$$g(x, \mu) = x \pm \mu x \pm x^3 \quad (8)$$

**Proof.** It is sufficient to observe that on one side of  $\mu = 0$ , any map in the family  $f_\mu$  has three transverse fixed points, one middle point  $x_M$  and two exterior fixed points  $x_L$  and  $x_R$  which have the opposite stability of  $x_M$ . Then,  $f_\mu$  and  $g_\mu$  share the same number of fixed points with the same stability and we apply Theorem 2.13. While, on the other side of  $\mu = 0$ , any map in the family  $f_\mu$  has only one transverse fixed point which is  $x_M$  with the same stability of 0 for  $g_\mu$ . Therefore, both maps verify Theorem 2.13. The same happens for  $f_0$  and  $g_0$ . For each one of those functions there exists a unique transverse fixed point with the same stability. ■

**Example 3.6** The Theorem 3.5 allows us to know that every family of any of the forms

$$x \pm \mu x \pm x^{2n+1} + h.o.t., \quad n \in \mathbb{N}$$

is locally topologically equivalent to one of the forms in (8).

**Theorem 3.7** (*Flip Normal Form*) Let  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a one-parameter family of maps verifying the generalized nondegeneracy conditions of a flip bifurcation [3]. Then,  $f_\mu$  is (locally) topologically equivalent near the origin to one of the following normal forms

$$g(x, \mu) = -x \pm \mu x \pm x^3 \quad (9)$$

**Proof.** It is sufficient to observe that on one side of  $\mu = 0$ , any map of the family  $f_\mu$  has the fixed point 0 and two 2-periodic points with the opposite stability of 0. Then,  $f_\mu$  and  $g_\mu$  satisfy the conditions of Theorems 2.17, 2.18 and Remark 2.20; while on the other side of  $\mu = 0$  any system  $f_\mu$  has only a fixed point which is 0 and both maps verify Theorems 2.19 and Remark 2.20. The same occurs for  $f_0$  and  $g_0$ . For each one of these maps there exists a unique fixed point with the same stability. ■

**Example 3.8** The Theorem 3.7 allows us to know that every family of any the forms

$$-x \pm \mu x \pm x^{2n+1} + h.o.t., \quad n \in \mathbb{N}$$

is locally topologically equivalent to one of the forms in (9).

**Remark 3.9** As is evident by the proofs, these (local) topological equivalences cover all the points that are relevant for the corresponding local bifurcations.

**Remark 3.10** It is worthwhile to observe that topological equivalences do not depend on the order of differentiability of the family  $f$ , which only must verify the corresponding bifurcation conditions.

## 4 Conclusions and future research directions

This paper shows how to find conjugacies which demonstrate the topological equivalence between any two increasing homeomorphisms with the same number of fixed points and with the same sequence of semi-stabilities and, for two decreasing homeomorphisms, both with one fixed point with no periodic orbits or both with a fixed point and one 2-period point.

As a consequence, these general results are applied to give a solution for a classical bifurcation problem which was set out several decades ago, but remained unpublished until now. Such a classical problem consists in finding topological conjugacies between families verifying specific bifurcation conditions and the simplest (truncated) normal form corresponding to the bifurcation. Actually, the problem is solved in more general way than originally posed, since the results work even for generalized bifurcation conditions independently of the order of differentiability of the families considered.

The topological conjugacies for families are *fiber* or *weak* in the sense stated in [2]. We leave as a future research work to demonstrate if it is possible to obtain the continuity of the homeomorphisms with respect to the parameter appearing in the families.

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